On Bertelson-Gromov Dynamical Morse Entropy

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Abstract

In this mainly expository paper we present a detailed proof of several results contained in a paper by M. Bertelson and M. Gromov on Dynamical Morse Entropy. This is an introduction to the ideas presented in that work.

Suppose M is compact oriented connected C^∞ manifold of finite dimension. Assume that $f_0 : M \to [0,1]$ is a surjective Morse function.

For a given natural number n, consider the set M^n and for $x = (x_0, x_1, ..., x_{n-1}) \in M^n$, denote $f_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f_0(x_j)$.

The Dynamical Morse Entropy describes for a fixed interval $I \subset [0,1]$ the asymptotic growth of the number of critical points of f_n in I, when $n \to \infty$.

The part related to the Betti number entropy does not requires the differentiable structure.

One can describe generic properties of potentials defined in the XY model of Statistical Mechanics with this machinery.

1 Introduction

We follow the main guidelines and notation of [7].

A Morse function is a smooth function such all critical points are not degenerate (see [16]).

Suppose M is compact oriented C^∞ manifold of dimension $q \geq 1$. Assume that $f_0 : M \to [0,1]$ is a surjective Morse function and Γ is a free group with basis $\gamma_1, ..., \gamma_n$. We assume that f_0 has p critical points ($p \geq 2$).

Suppose $\Omega \subset \Gamma$ is a finite non-empty set. If $x \in M^\Omega$ we denote $x_\gamma \in M$, $\gamma \in \Omega$, the corresponding coordinate.

Then, we define $f_\Omega : M^\Omega \to [0,1]$ by the expression

$$f_\Omega(x) = \frac{1}{|\Omega|} \sum_{\gamma \in \Omega} f_0(x_\gamma),$$

where $|\Omega|$ is the cardinality of Ω. This function f_Ω is also a surjective Morse function.

2 The XY model

As a particular case we can consider $\Gamma = \mathbb{Z}$, the set $M^{\mathbb{Z}}$ and for $x = (x_j)_{j \in \mathbb{Z}} \in M^{\mathbb{Z}}$, $n > 0$, $f_0 : M \to \mathbb{R}$, and

$$f_n(x) = -\frac{1}{n} \sum_{j=0}^{n-1} f_0(x_j).$$

The question about the minus sign in front of the sum is not important but if we want that f_0 represents a kind of energy we will keep the $-$ (at least in this section).

In the model it is natural to consider that adjacent molecules in the lattice interact via a potential (energy) which is described by the smooth function of two variables f_0. The mean energy up to position n is described by f_n. The points $x \in M^n$ where the mean n-energy is lower or higher are of special importance. We are interested here, among other things, in the growth of the number of critical values, when $n \to \infty$. The critical points are called the stationary states (see [7]).

Denote by $\mathrm{Cri}_n(I)$ the number of critical points of f_n in a certain interval $f^{-1}(I)$. Roughly speaking the purpose of [7] is to provide for a fixed value $c \in [0,1]$ a topological lower bound for

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{\log(\mathrm{Cri}_n(I))}{n}, \text{ where } I = (c - \delta, c + \delta),$$

in terms of a certain strictly positive concave function (a special kind of entropy). This is done by taking into account the homological behavior of the functions f_n.

The so called classical XY model consider the case where $M = S^1$ (see for instance [2], [6], [5], [10], [20], [13], [9] or [19]). A function $A : (S^1)^{\mathbb{Z}} \to \mathbb{R}$ describes interaction between sites on the lattice $\mathbb{Z}$ where the spins are on S^1. One is interested in equilibrium probabilities $\hat{\mu}$ on $(S^1)^{\mathbb{Z}}$ which are invariant for the shift $\hat{\sigma} : (S^1)^{\mathbb{Z}} \to (S^1)^{\mathbb{Z}}$. A point x on $(S^1)^{\mathbb{Z}}$ is denoted by $x = (..., x_{-2}, x_{-1} \,|\, x_0, x_1, x_2, ...)$.

In the case the potential A depend just on the first coordinate $x_0 \in S^1$, that is $A(x) = f_0(x_0)$, then the setting described above applies.

In the case the potential A depend just on the two first coordinate $x_0, x_1 \in S^1$, that is $A(x) = f_0(x_0, x_1)$, then, we claim that the setting described above in the introduction applies. This is the case when $f_0 : S^1 \times S^1 \to \mathbb{R}$. Indeed, in this case one can take $M = S^1 \times S^1$ and consider that f_0 acts on M. In this case we can say that f_0 depends just in the first coordinate on $M^{\mathbb{Z}} = (S^1 \times S^1)^{\mathbb{Z}}$ and adapt the general formalism we describe here.

Therefore, we will state all results for $f_0 : M \to \mathbb{R}$, that is, the case the potential on $M^{\mathbb{Z}}$ depends just on the first coordinate.

In the case $\hat{\mu}$ is ergodic f_n describes Birkhoff means which are $\hat{\mu}$ almost everywhere constant. We are here interested more in the topological and not in the measure theoretical point of view.

In the measure theoretical (or Statistical Mechanics) point of view, if one is interested in equilibrium states at positive temperature $T = 1/\beta$, then, is natural to consider expressions like $\int e^{\sum_{j=0}^{n-1} -\beta\, f_0(x_j)}\, dx_0\, dx_1 ... d_{x_{n-1}}$ (or, when the set of spins is finite: $\sum e^{\sum_{j=0}^{n-1} -\beta\, f_0(x_j)}$) and its normalization (see [18], [11] and [12]) which defines the partition function.

By the other hand if one is interested in the zero temperature case (see for instance [4]), then, expressions like $-\sum_{j=0}^{n-1} f_0(x_j)$ are the main focus. For instance, if f_0 has a unique point of minimum $x^- \in S^1$, then $\delta_{(x^-)^\infty}$ defines the ground state (maximizing probability). In the generic case the function f_0 has indeed a unique point of minimum.

Given $f_0 : M \times M \to \mathbb{R}$ and n one can also consider periodic conditions. In this case we are interested in sums like

$$\tilde{f}_n(x) = -\frac{1}{n}\,(f_0(x_0) + f_0(x_1) + ... + f_0(x_{n-2}) + f_0(x_0)),$$

or

$$-(f_0(x_0) + f_0(x_1) + ... + f_0(x_{n-2}) + f_0(x_0)).$$

In the case we want to get Gibbs states via the Thermodynamic Limit (see for instance [11] or [18]), given a natural number n, we have to look for the probability μ on M^n (absolutely continuous with respect to Lebesgue probability) which maximizes

$$\int e^{-\sum_{j=0}^{n-1} \beta\, f_0(x_j)}\, d\,\mu(dx_0,\, dx_1, ..., d_{x_{n-1}}),$$

or, at zero temperature the periodic probability μ on M^n which maximizes

$$-\int \sum_{j=0}^{n-1} f_0(x_j)\; d\,\mu(dx_0,\, dx_1, ..., d_{x_{n-1}}).$$

In the last case when there is a unique point x^- of minimum for f_0 then for each n the solution μ is a delta Dirac on $(x^-)^n$.

One can easily adapt the reasoning of [1] to show that for a generic f_0 we get that $\tilde{f}_n$ is a Morse function for all n.

When f_0 is not generic several pathologies can occur (see for instance [19], [2] and [10]).

Suppose the case when there is a unique point x^- of minimum for f_0. For each $\beta > 0$ and n denote by $\mu_{n,\beta}$ the absolutely continuous with respect to Lebesgue probability which maximizes

$$\int e^{-\sum_{j=0}^{n-1} \beta f_0(x_j)} d\,\mu(dx_0,\, dx_1, ..., d_{x_{n-1}}).$$

By the Laplace method (adapting Proposition 3 in [5] or Lemma 4 in [6]) we get that when $\beta \to \infty$ and $n \to \infty$ the probability $\mu_{n,\beta}$ converges to the Dirac delta on $(x^-)^\infty$. Therefore, in the generic case this last probability is the ground state (zero temperature limit).

3 The general model - the dynamical Morse entropy

From now we forget the $-$ sign in front of f_0. For instance, $f_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f_0(x_j, x_{j+1})$.

Given $c \in [0,1]$ and $\delta > 0$, take $N_\Omega(c,\delta)$ the number of critical points of f_Ω in $f_\Omega^{-1}[c-\delta, c+\delta]$. Note that if f_0 has p critical points then f_Ω has $p^{|\Omega|}$ critical points.

Consider the cylinder sets

$$\Omega_i \,=\, \{a_1\,\gamma_1 + ... + a_n\gamma_n\,;\, |a_j| \leq i,\, 1 \leq\, j \leq n\,\},\; i = 1, 2, ...$$

Denote $N_i(c,\delta) = N_{\Omega_i}(c,\delta)$. Then, of course, $N_i(c,\delta)$ for c fixed decrease with δ.

For a fixed $0 \leq c \leq 1$, we denote the entropy by

$$\epsilon(c) = \lim_{\delta\to 0}\,(\,\liminf_{i\to+\infty}\,\frac{\log(N_i(\,c,\delta)\,)}{|\Omega_i|}\,).$$

The above limit exists is bounded by $\log p$ but in principle could take the value $-\infty$. We call $\epsilon(c)$ the **dynamical Morse entropy** on the value c.

In the case $\Gamma = \mathbb{Z}$ as we mentioned before $\epsilon(c)$ is described by

$$\epsilon(c) = \lim_{\delta\to 0} \,(\,\liminf_{n\to+\infty} \frac{\log(\text{number of critical points of } f_n \text{ in } f_n^{-1}[c-\delta, c+\delta]\,)}{n}\,).$$

Later we introduce a function $b(c)$ (see Definition 11 and also Definition 4), which will be a topological invariant of f_0. The function $b(c)$ is defined in terms of rank of linear operators and Cohomology groups.

We will show later that

1) $0 \leq b(c) \leq \epsilon(c),\ 0 \leq c \leq 1$;

2) $b(c)$ is continuous and concave;

3) $b(c)$ is not constant equal to 0.

Finally, in the case $M = S^1$ (the unitary circle) and f_0 has just two critical points, we show in section 7 that

$$\epsilon(c) = b(c) = -c\log c - (1-c)\log(1-c).$$

$b(c)$ is sometimes called the **Betti entropy** of f_0.

Our definition of $b(c)$ is different from the one in [7] but we will show later (see section 8) that is indeed the same.

A key result in the understanding of the main reasoning of the paper is Lemma 9 which claims that for any Morse function f, given $a, b \in \mathbb{R}$, $a < b$, the number of critical points of f in $f^{-1}[a,b]$ is bigger or equal to the dimension of the vector space

$$\frac{H^*(f^{-1}(\infty, b)\,)}{H^*(f^{-1}(-\infty, a)\,)},$$

where H^* denotes the corresponding cohomology groups which will be defined in the following paragraphs (see also [15] for basic definitions and properties).

$H^*(X, \mathbb{R})$ denotes the usual cohomology. Note that H^* will have another meaning (see definition 1).

4 Cohomology

Suppose X is a metrizable, compact, oriented topological manifold C^∞ manifold. We will consider the singular homology. Suppose $U \subset X$ is an open set and $a \in H^*(X, \mathbb{R})$. The meaning of the statement supp $a \subset U$ is: there exist an open set $V \subset X$, such that, $X = U \cup V$, and $a|_V = 0$.

Definition 1. *$H^*_X(U) = \{\, a \in H^*(X, \mathbb{R}) \;:\; supp \;\; a \subset U\}$, where U is an open subset of X. When X is fixed we denote $H^*_X(U) = H^*(U)$.*

Remember (see for instance [15]) that when $U \subset X$ is open we get the exact cohomology sequence:

$$... \to H^{k-1}(X-U,\mathbb{R}) \to H^k_c(U,\mathbb{R}) \to H^k(X,\mathbb{R}) \to H^k(X-U,\mathbb{R}) \to H^{k+1}_c(U,\mathbb{R}) \to ... \tag{1}$$

where H^*_c denotes the support compact cohomology.

Lemma 2. *If U is an open set, then*

$$H^*(U) = Im(\ H^*_c(U,\mathbb{R}) \to H^*(X,\mathbb{R})\ ) = \ Ker(\ H^*(X,\mathbb{R}) \to H^*(X-U,\mathbb{R})\ ).$$

Proof: The second equality follows from the fact that the above sequence is exact.

We will prove that

$$\mathrm{Im}(\ H^*_c(U,\mathbb{R}) \to H^*(X,\mathbb{R})\ ) \subset H^*(U) \subset \mathrm{Ker}\,(\ H^*(X,\mathbb{R}) \to H^*(X-U,\mathbb{R})\ ).$$

Let $a \in \mathrm{Im}(\ H^*_c(U,\mathbb{R}) \to H^*(X,\mathbb{R})\ )$. Then, a is represented by a cocycle α with compact support $K \subset U$. Therefore, $a\,|(X-K) = 0$.

Defining $V = X - K$ we have that $U \cup V = X$ and $a\,|V = 0$. Then, $a \in H^*(U)$.

Let be $\alpha \in H^*(U)$. Let $V \subset X$ be an open set such that $U \cup V = X$ and $\alpha\,|V = 0$.

Since $X - U \subset V$, we have $\alpha\,|(X-U) = 0$.

Then, $\alpha \in \mathrm{Ker}\,(\,H^*(X,\mathbb{R}) \to H^*(X-U,\mathbb{R})\,)$. □

Lemma 3. *If U is an open set then $H^*(U)$ is a graded ideal of the ring of cohomology of X.*

Proof: This follows at once from Lemma 2.

□

Now we consider a continuous function $f : X \to \mathbb{R}$.

Definition 4. *Given $\delta > 0$ and $c \in \mathbb{R}$ we define*

$$b'_{c,\delta} = Dim\,(\,\frac{H^*(f^{-1}(-\infty, c+\delta)\,)\,)}{H^*(f^{-1}(-\infty, c-\delta)\,)}\,).$$

Proposition 5. *Suppose X and Y are metrizable compact, oriented topological manifolds, moreover take $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ continuous functions. If we define $f \oplus g : X \times Y \to \mathbb{R}$, by $(f \oplus g)(x,y) = f(x) + g(y)$, then, if $c, c' \in \mathbb{R}$, $\delta, \delta' > 0$, we get*

$$b'_{c,\,\delta}(f)\ b'_{c',\,\delta'}(g)\ \leq b'_{c+c',\,\delta+\delta'}(f \oplus g). \tag{2}$$

Before the proof of this import proposition we need two more lemmas.

As it is known (see [15]) the cup product $\vee$ defines an isomorphism

$$\mu : H^*(X,\mathbb{R}) \otimes H^*(Y,\mathbb{R}) \;\rightarrow\; H^*(X \times Y,\mathbb{R}).$$

Lemma 6. *If $U \subset X$ and $V \subset Y$ are open sets, then*

$$\mu(\, H^*_X(U) \otimes H^*(Y,\mathbb{R}) + H^*(X,\mathbb{R}) \otimes H^*_Y(V)\,) \;= H^*_{X\times Y}(\,(U \times Y) \cup (X \times V)\,).$$

Proof: By Lemma 2 we get

$$H^*_{X\times Y}(\,(U\times Y)\cup(X\times V)\,) = \mathrm{Ker}\,(\,H^*(X\times Y,\mathbb{R}) \rightarrow H^*((X-U)\times(Y-V),\mathbb{R}).$$

Then,

$$H^*_{X\times Y}(\,(U \times Y) \cup (X \times V)\,) =$$
$$\mu(\;\mathrm{Ker}\,(\,H^*(X,\mathbb{R}) \otimes H^*(Y,\mathbb{R}) \;\rightarrow H^*(X-U,\mathbb{R}) \otimes H^*(Y-V,\mathbb{R})\,)\;).$$

From simple Linear Algebra arguments the claim follows from Lemma 2.

□

Lemma 7. *If $U \subset X$ and $V \subset Y$ are open sets then*

$$\mu(\,H^*_X(U) \otimes H^*_Y(V)\,) = H^*_{X\times Y}(U \times V).$$

Proof: The $\vee$ product defines a natural isomorphism

$$H^*(X,X-U,\mathbb{R})\otimes H^*(Y,Y-V,\mathbb{R}) \;\rightarrow H^*(X\times Y,(X\times(Y-V)\cup(X-U)\times Y,\mathbb{R}) =$$

$$H^*(\,X \times Y,(X \times Y) - (U \times V,\mathbb{R})\,).$$

By Lemma 2 and the exact relative cohomology sequence we get:

$$H^*_X(U) = \mathrm{Im}\,(\,H^*(X,X-U,\mathbb{R}) \;\rightarrow H^*(X,\mathbb{R})\,),$$

$$H^*_Y(V) = \mathrm{Im}\,(\,H^*(Y,Y-V,\mathbb{R}) \;\rightarrow H^*(Y,\mathbb{R})\,),$$

and

$$H^*_{X\times Y}(U \times V) = \mathrm{Im}\,(\,H^*(X \times Y,(X \times Y) - (U \times V,\mathbb{R})) \;\rightarrow\; H^*(X \times Y,\mathbb{R})\,).$$

From this the claims follows at once.

□

Now we will present the proof of Proposition 5.

Take $h = f \oplus g$ and denote

$$A^- = f^{-1}(-\infty, c{-}\delta),\ \ B^- = g^{-1}(-\infty, c'{-}\delta'),\ \ C^- = h^{-1}(-\infty, (c{+}c'){-}(\delta{+}\delta'))\ ,$$

and

$$A^+ = f^{-1}(-\infty, c{+}\delta),\ \ B^+ = g^{-1}(-\infty, c'{+}\delta'),\ \ C^+ = h^{-1}(-\infty, (c{+}c'){+}(\delta{+}\delta'))\ .$$

Note that

$$A^+ \times B^+ \subset C^+ \subset (A^+ \times Y) \cup (X \times B^+)$$

$$A^- \times B^- \subset C^- \subset (A^- \times Y) \cup (X \times B^-).$$

Consider the commutative diagram

$$\begin{array}{ccc} H^*(X,\mathbb{R}) \otimes H^*(Y) & \to (\text{using}\, \mu\,) & H^*(X \times Y, \mathbb{R}) \\ \cup & & \cup \end{array}$$

$$H^*_X(A^+) \otimes H^*_Y(B^+) \to H^*_{X\times Y}(C^+) \subset H^*_{X\times Y}(\,(A^+ \times Y) \cup (X \times B^+)\,)$$

$$\cup \qquad\qquad\qquad\qquad\qquad\qquad \cup$$

$$H^*_X(A^+) \otimes H^*_Y(B^-) + H^*_X(A^-) \otimes H^*_Y(B^+) \to H^*_{X\times Y}(\,(A^- \times Y) \cup (X \times B^-)\,)$$

$$\cup \qquad\qquad \cup$$

$$H^*_X(A^-) \otimes H^*_Y(B^-) \to H^*_{X\times Y}(C^-).$$

From this follows the linear transformation

$$\tilde{\mu} :\ \frac{H^*_X(A^+) \otimes H^*_Y(B^+)}{H^*_X(A^-) \otimes H^*_Y(B^-)} \to \frac{H^*_{X\times Y}(C^+)}{H^*_{X\times Y}(C^-)}.$$

By the other hand

$$(\,H^*_X(A^+) \otimes H^*_Y(B^+) \cap \mu^{-1}(\,H^*_{X\times Y}(C^-)\,) \subset$$

$$(\,H^*_X(A^+) \otimes H^*_Y(B^+) \cap \mu^{-1}(\,H^*_{X\times Y}(\,(A^- \times Y) \cup (X \times B^-)\,)\,) =$$

$$(\,H^*_X(A^+) \otimes H^*_Y(B^+)\,) \cap (\,H^*_X(A^-) \otimes H^*(Y, \mathbb{R}) + H^*(X, \mathbb{R}) \otimes H^*_Y(B^-)\,) =$$

$$H^*_X(A^-) \otimes H^*_Y(B^+)\ +\ H^*_X(A^+) \otimes H^*_Y(B^-)\,.$$

The first equality above follows from Lemma 6; the second follows from Linear Algebra; namely, if $E_2 \subset E_1 \subset E$ and $F_2 \subset F_1 \subset F$, then

$$(E_1 \otimes F_1) \cap (E_2 \otimes F + E \otimes F_2) = E_2 \otimes F_1 + E_1 \otimes F_2.$$

From the above it follows that

$$\operatorname{Ker} \tilde{\mu} \subset \frac{H^*_X(A^-) \otimes H^*_Y(B^+) \, + \, H^*_X(A^+) \otimes H^*_Y(B^-)}{H^*_X(A^-) \otimes H^*_Y(B^-)}.$$

Therefore,

$$\begin{aligned} b'_{c+c',\delta+\delta'} \; = \; \dim \frac{H^*_{X\times Y}(C^+)}{H^*_{X\times Y}(C^-)} \geq \dim (\operatorname{Im} \tilde{\mu}) \geq \\ \dim \frac{H^*_X(A^+) \otimes H^*_Y(B^+)}{H^*_X(A^-) \otimes H^*_Y(B^+) \, + \, H^*_X(A^+) \otimes H^*_Y(B^-)} = \\ \dim \Big(\frac{H^*_X(A^+)}{H^*_Y(A^-)} \otimes \frac{H^*_Y(B^+)}{H^*_Y(B^-)} \Big) = b'_{c,\delta}(f) \; b'_{c',\delta'}(g). \end{aligned}$$

□

5 Critical points

In what follows X is a compact, oriented C^∞ manifold and $f : X \to \mathbb{R}$ is a Morse function.

Lemma 8. *Suppose X is a compact, oriented C^∞ manifold and $U \subset X$ is an open set. If $a \in H^*(X, \mathbb{R})$, then, supp $a \subset U$, if and only if, there exists a closed C^∞ differentiable form w such that supp $w \subset U$, and a is the de Rham cohomological class of w.*

Proof: If there exists $w \in a$, such that supp $w \subset U$, then

$$a|_{(X - \text{supp } w)} = 0 \,\text{and}\; U \,\cup\, (X - \text{supp } w) \;=\; X.$$

If there exists an open set $V \subset X$ such that $U \cup V = X$ and $a|_V = 0$, then, there exist a C^∞ form η on V such that $d\eta = w|_V$ where $w \in a$.

Let W be an open set such that $\overline{W} \subset V$ and $W \cup U = X$. Take a C^∞ function $\varphi : X \to [0,1]$ such that $\varphi|_{\overline{W}} = 1$ and $\varphi|_{X-K} = 0$, where K is compact set such that $\overline{W} \subset K \subset V$. Then, $\varphi\,\eta$ has an extension to X and $(w - d\,(\varphi\,\eta)\,) \in a$. But,

$$\text{supp} \;\; (w - d\,(\varphi\,\eta)\,) \subset X - W \subset U.$$

□

Lemma 9. *Given $a, b \in \mathbb{R}$, $a < b$, then, the number of critical points of f in $f^{-1}[a, b]$ is bigger or equal that*

$$dim\ \frac{H^*(f^{-1}(\infty, b)\,)}{H^*(f^{-1}(-\infty, a)\,)}.$$

Proof:

Without lost of generality we can assume that a and b are regular values of f (decrease a and increase b a little bit).

Given $c_1 < c_2 < ... < c_m$, the critical values of f in (a, b), take

$$a = d_0 < c_1 < d_1 < c_2 < d_2 < ... < d_{m-1} < c_m < d_m = b.$$

By proposition 16 and Lemma 14, the number of critical points in $f^{-1}(c_i)$, $i = 1 =, 2, ..., m$, is bigger or equal to

$$dim\ \frac{H^*(f^{-1}(\infty, d_i)\,)}{H^*(f^{-1}(-\infty, d_{i-1})\,)}.$$

Finally consider the filtration

$$H^*(f^{-1}(\infty, a)\ =\ H^*(f^{-1}(-\infty, d_0)\,) \subset\ H^*(f^{-1}(-\infty, d_1)\,) \subset ...$$

$$\subset\ H^*(f^{-1}(-\infty, d_{m-1})\ \subset\ H^*(f^{-1}(-\infty, d_m)\,) =\ H^*(f^{-1}(-\infty, b)\,).$$

$\square$

Now we denote $b'_\Omega(c, \delta) = b'_{c,\delta}(f_\Omega)$ and $b'_i(c, \delta) = b'_{\Omega_i}(c, \delta)$, $0 \leq c \leq 1$, $\delta > 0$.

Corollary 10. $b'_i(c, \delta) \leq N_i(c, \delta)$ *for all* $i = 1, 2, 3, ...$ *and* $0 \leq c \leq 1$, $\delta > 0$.

Now we define the function b using Proposition 16 a)

Definition 11.

$$b(c) = \lim_{\delta \to 0} \liminf_{i \to \infty} \frac{\log(b'_i(c, \delta))}{|\,\Omega_i\,|},\ \ 0 \leq c \leq 1.$$

We will show that in above definition we can change the lim inf by lim.

Lemma 12.

$$b(c) \leq \epsilon(c) \leq \log(\ \textit{the number of critical points of}\ \ f_0\,).$$

Proof: The first inequality follows from corollary 10. From the definition is easy to see that $\epsilon(c)$ is smaller than log of the number of critical points of f_0.

□

We denote $B(\Gamma)$ a family of finite subsets of Γ and $B_N(\Gamma)$, $N \in \mathbb{N}$, the family of sets $\Omega \in B(\Gamma)$ such that $|\Omega| > N$.

Proposition 13. *Suppose $\Omega', \Omega'' \in B(\Gamma)$ are disjoint not empty sets. Then,*

$$b'_{\Omega\cup\Omega''}(\alpha c_1 + (1-\alpha)c_2, \delta) \geq b'_{\Omega'}(c_1, \delta)\, b'_{\Omega''}(c_2, \delta),$$

where $0 \leq c_1, c_2 \leq 1$, $\delta > 0$ *and* $\alpha = \frac{|\Omega'|}{|\Omega'|+|\Omega''|}$.

Proof:
By definition

$$f_{\Omega'\cup\Omega''} = \alpha f_{\Omega'} \oplus (1-\alpha) f_{\Omega''}.$$

By Proposition 5, as $\delta = \alpha\,\delta + (1-\alpha)\delta$, then

$$b'_{\alpha\, c_1 + (1-\alpha)\, c_2, \delta}(f_{\Omega'\cup\Omega''}) \geq b'_{\alpha\, c_1, \alpha\, \delta}(\alpha\, f_{\Omega'})\, b'_{(1-\alpha)\, c_2, (1-\alpha)\, \delta}((1-\alpha)\, f_{\Omega''}) =$$

$$b'_{c_1,\delta}(f_{\Omega'})\ \ b'_{c_2,\delta}(f_{\Omega''}).$$

□

Lemma 14. *Suppose the interval $[a, b]$ does no contains critical values of f. Then,*

$$H^*(\, f^{-1}(\, -\infty, a\,)\,) = H^*(\, f^{-1}(\, -\infty, b\,)\,).$$

Proof: This follows from Lemma 2 and the fact that $f^{-1}[b,\, \infty)$ is a deformation retract of $f^{-1}[a,\, \infty\,)$.

□

Definition 15. *Given $c \in \mathbb{R}$ we define*

$$\tilde{b}_c(f) = \lim_{\delta\to 0} b'_{c,\delta}(f).$$

Proposition 16. *For a fixed c we have*

a) $b'_{c,\delta}(f)$ decreases with δ and $b'_{c,\delta}(f) = \tilde{b}_c(f)$ for all δ small enough.

b) $\tilde{b}_c(f) = 0$ if c is not a critical value of f

c) $\tilde{b}_c(f)$ is smaller than the number of critical points of f in $f^{-1}(c)$

d) $\sum_c \tilde{b}_c(f) = Dim\ H^*(X)$.

Proof:

a) follows from the above definitions and Lemma 14.

b) follows from Lemma 14

For the proof of c) consider the exact diagram

$$\begin{tikzcd}
 & H^*(X,\mathbb{R}) \arrow[d, "r_1"] \arrow[dr, "r_2"] & \\
H^*(f^{-1}[c{-}\delta,\infty)), f^{-1}(c{+}\delta,\infty), \mathbb{R}\,) \arrow[r] & H^*[f^{-1}(c{-}\delta,\infty), \mathbb{R}) \arrow[r] & H^*(f^{-1}[c{+}\delta,\infty), \mathbb{R}),
\end{tikzcd}$$

where r_1 and r_2 are the restriction homomorphisms.

By lemma 2

$$H^*(f^{-1}(-\infty, c+\delta)) = \mathrm{Ker}\, r_2 \quad \text{and} \quad H^*(f^{-1}(-\infty, c-\delta)) = \ \mathrm{Ker}\, r_1.$$

From this follows that

$$b'_{c,\delta}(f) = \ \mathrm{Dim}\ (r_1(\mathrm{Ker}\,(r_2)\,) \ \leq \ \mathrm{Dim}\,(\,H^*(f^{-1}(c-\delta,\infty)), f^{-1}(c+\delta,\infty)\,), \mathbb{R}\,)$$

because the above sequence is exact.

In order to finish the proof we apply Morse Theory (see [16]) with δ small enough.

For the proof of d) suppose $c_1 < c_2 < ... < c_m$ are the critical values of of f. Now, consider

$$d_0 < c_1 < d_1 < c_2 < d_2 < ... < d_{m-1} < c_m < d_m.$$

Now, from a) and Lemma 14 we have

$$\tilde{b}_{c_i}(f) = \mathrm{Dim}\,(\frac{H^*(\,f^{-1}(-\infty, d_i)\,)}{H^*(\,f^{-1}(-\infty, d_{i-1})\,)}\,), \quad i = 1, 2, ..., m.$$

Finally, note that

$$0 = H^*(\,f^{-1}(-\infty, d_0)) \subset H^*(\,f^{-1}(-\infty, d_1)) \subset ... \subset$$

$$H^*(\,f^{-1}(-\infty, d_m)) = H^*(X).$$

$\square$

Lemma 17. *Given $\delta > 0$, there exists an integer N such that: $b'_\Omega(c,\delta) \geq 1$ for all $c \in [0,1]$ and all $\Omega \in B_N(\Gamma)$. Therefore, $b(c) \geq 0$, for all $0 \leq c \leq 1$.*

Before the proof of lemma 17 we need two more lemmas.

Lemma 18. *Suppose X is a compact oriented C^∞ manifold and $f : X \to \mathbb{R}$ is a Morse function. Then, for all $\delta > 0$*

$$b'_{a_1,\delta}(f) \geq 1 \text{ and } b'_{a_2,\delta}(f) \geq 1,$$

where a_1 and a_2 are respectively the maximum and minimum of f.

Proof: If δ is small enough, $f^{-1}(-\infty, a_2 + \delta)$ is the disjoint union of a finite number of open discs and $f^{-1}(-\infty, a_2 - \delta) = \emptyset$.

If n is the dimension of X, then, it follows from Lemma 2 that

$$H^n(X, \mathbb{R}) \subset H^*(f^{-1}(-\infty, a_2 + \delta)\,) \neq 0$$

and

$$H^*(f^{-1}(-\infty, a_2 - \delta)\,) = 0.$$

Then, $b'_{a_2,\delta}(f) \geq 1$, if $\delta > 0$ is small enough. Therefore, this claim is also true for any $\delta > 0$ by Proposition 16 a).

In a similar way we have that for small $\delta > 0$

$$H^0(X, \mathbb{R}) \subset H^*(f^{-1}(-\infty, a_1 + \delta)\,)$$

and

$$H^0(X, \mathbb{R}) \text{ is not contained } H^*(f^{-1}(-\infty, a_1 - \delta)\,).$$

From this the final claim is proved.

□

Lemma 19. *Consider $\Omega \in B(\Gamma)$ where $|\Omega| = m \geq 1$, then, $b'_\Omega(k/m,\, \delta) \geq 1$, for all $\delta > 0$ and $k = 0, 1, 2, ..., m$.*

Proof: If $k = 0$, or m, the claim follows from Lemma 18 with $X = M^\Omega$, $f = f_\Omega$.

Given $0, k, m$, $0 < k < m$, take $\Omega = \Omega' \cup \Omega''$, where Ω', Ω'' are disjoints and $k = |\Omega'|$.

By Proposition 13 with $c_1 = 1$ and $c_2 = 0$ we get

$$b'_\Omega(k/m,\, \delta) \geq b'_{\Omega'}(1,\, \delta)\, b'_{\Omega''}(0,\, \delta) \geq 1.$$

Yet from last lemma.

□

Now we will prove Lemma 17.

Proof:

Take $N > \frac{2}{\delta}$, $\Omega \in B_N(\Gamma), |\Omega| = m > N$ and k such that $\frac{k}{m} \leq c < \frac{k+1}{m}$,

By definition,

$$b'_{c,\delta}(f_\Omega) \geq b'_{k/m,\,\delta/2}(f_\Omega),$$

since $c - \delta < k/m - \delta/2$ and $c + \delta > k/m + \delta/2$.

Therefore, $b'_\Omega(c, \delta) \geq b'_\Omega(k/m,\, \delta/2) \geq 1$ by Lemma 19.

□

Proposition 20.

$$0 \leq b(c) \leq \epsilon(c) \leq \log(\textit{number of critical points of} f_0\,),\ \ 0 \leq c \leq 1.$$

Proof: This follows from Lemma 12 and Lemma 17

□

Lemma 21. *Given $c \in [0, 1]$ and $\delta > 0$, consider a non-empty set $\Omega \in B(\Gamma)$ and $\gamma \in \Gamma$. Then,*

$$b'_\Omega(c, \delta) = b'_{\Omega+\gamma}(c, \delta).$$

In the case $\Gamma = \mathbb{Z}$ we have that for any $\Omega = \{1, 2, ..., k\}$

$$b'_\Omega(c, \delta) = b'_{\hat{\sigma}(\Omega)}(c, \delta),$$

where $\hat{\sigma}$ is the shift acting on $M^{\mathbb{Z}}$.

Proof: For fixed γ consider the transformation $x \in M^\Omega \to y \in M^{\Omega+\gamma}$, such that $y_w = x_{w-\gamma}$, which is a diffeomorphism which commutes $f_{\Omega+\gamma}$ with f_Ω.

The result it follows from this fact.

□

We will show now that indeed one can change lim inf by inf in Definition 11. In order to do that we need the following proposition which describes a kind of subadditivity.

Proposition 22. *Given an integer number $N > 0$ take $h : B_N(\Gamma) \to \mathbb{R}$, $h \geq 0$, which is invariant by Γ and such that*

$$h(\Omega' \cup \Omega'') \geq h(\Omega') + h(\Omega''),$$

if Ω', $\Omega'' \in B_N(\Gamma)$, are disjoint. Then, there exists

$$\lim_{i\to\infty} \frac{h(\Omega_i)}{|\Omega_i|} \geq 0,\ \ (\textit{finite or} +\infty\,).$$

From this follows:

Corollary 23. *For $c \in [0,1]$ and $\delta > 0$,*

a) there exist the limit

$$\lim_{i\to\infty} \frac{\log b'_i(c,\delta)}{|\,\Omega_i\,|} \;=\; b'(c,\delta).$$

b) $0 \le b'(c,\delta) \le \log($ *number of critical points of* f_0 *),*

c) $b(c) = \lim_{\delta\to 0} b'(c,\delta)$

Proof: The claim a) follows from last proposition applied to $h(\Omega) = \log b'_\Omega(c,\delta)$, by Lemma 17, Proposition 13 taking $c_1 = c_2 = c$ and also by Lemma 21.

Item b) follows from lemma 13 and corollary 10.

Item c) follows from item a) and the definition of $b(c)$.

□

Before the proof of Proposition 22 we need two lemmas.

Lemma 24. *Given an integer positive number k, then for each $i > (3\,k+1)$ there exists $\Omega_{k,i} \in B(\Gamma)$ such that: a) $\Omega_{k,i} \subset \Omega_i$; b) $\Omega_{k,i}$ is a disjoint union of a finite number of translates of Ω_k ; c) $\lim_{i\to\infty} \frac{|\,\Omega_{k,i}|}{|\,\Omega_i\,|} = 1$; d) $|\Omega_i| - |\Omega_{k,i}| \ge (2k+1)^n$, where n is the number of generators of Γ.*

Proof: For the purpose of the proof we can assume that $\Gamma = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{n}$ and take $\gamma_1, \gamma_2, .., \gamma_n$ the canonical basis.

Take $m \ge 1$ an integer such that

$$k + \,m\,(2\,k + \,1) \le i < k = \,(m+1)\,(2\,k+1),$$

and

$$\Omega_{k,i} = \cup\ \{\ \Omega_k + (\,j_1(2k+1), ..., j_n(2k+1)\,)\,|$$
$$-m \le j_1, .., j_n \le m,\ (j_1, ..., j_n) \ne (0, ..., 0)\ \}.$$

It is easy to see that the sets $\Omega_{k,i}$ satisfy all the above claims.

□

Lemma 25. *Given real numbers $x_i \ge 0$ $i = 1, 2, 3, ...,$ suppose that for each k and each $\epsilon > 0$ there exist $N_{k,\epsilon}$ such that*

$$x_i \ge x_k(1-\epsilon) \text{ } if \text{ } i \ge N_{k,\epsilon}.$$

Then, there exists $\lim_{i\to\infty} x_i$ (which is finite or $+\infty$).

Proof: Take $L = \limsup_{i\to\infty} x_i$ and $a \in \mathbb{R}$, $a < L$. Then, there exists $x_k > a$. Therefore, $x_i \geq a$, if i is very large. Then, $\liminf_{i\to\infty} x_i \geq a$. From this follows the claim.

$\square$

Now we will prove Proposition 22.

Proof: Suppose k is such that $(2k+1)^n > N$. Take $i > 3\,k+1$, then, $|\,\Omega_{k,i}\,| \geq (2k+1)^n > N$ and $|\Omega_i - \Omega_{k,i}| \geq (2k+1)^n > N$.
Then, $h(\Omega_i) = h(\,\Omega_{k,i} \cup (\,\Omega_i - \Omega_{k,i})\,) \geq h(\Omega_{k,i})$.
Moreover, each translate of Ω_k has cardinality $(2k+1)^n$. Therefore,

$$h(\Omega_{k,i}) \;\geq\; \frac{|\,\Omega_{k,i}\,|}{|\,\Omega_k\,|} h(\Omega_k).$$

From this follows that

$$\frac{h(\Omega_i)}{|\,\Omega_i\,|} \geq \frac{h(\Omega_{k,i})}{|\,\Omega_{k,i}\,|}\,\frac{|\,\Omega_{k,i}\,|}{|\,\Omega_i\,|} \geq \frac{h(\Omega_k)}{|\,\Omega_k\,|}\,\frac{|\,\Omega_{k,i}\,|}{|\,\Omega_i\,|},$$

and the claim is a consequence of Lemmas 24 and 25.

$\square$

The next lemma will be used later

Lemma 26. *Under the hypothesis of Proposition 22 consider*

$$\Omega_i' = (\,\Omega_i + (2i+1)\,\gamma_1\,) \cup \Omega_i, \;\; i = 1,2,3,...$$

Then,

$$\lim_{i\to\infty} \frac{h(\Omega_i')}{|\,\Omega_i'\,|} = \lim_{i\to\infty} \frac{h(\Omega_i)}{|\,\Omega_i\,|}.$$

Proof: If $i > N$, then $|\Omega_i| > N$. Therefore,

$$h(\Omega_i') \geq h(\,\Omega_i + \;(\,2\,i+1\,)\,\gamma_1\,) \;+\; h(\Omega_i) = 2h(\Omega_i).$$

From this follows

$$\frac{h(\Omega_i')}{|\,\Omega_i'\,|} \;\geq\; \frac{h(\Omega_i)}{|\,\Omega_i\,|}.$$

Therefore,

$$\liminf_{i\to\infty} \frac{h(\Omega_i')}{|\,\Omega_i'\,|} \;\geq\; \liminf_{i\to\infty} \frac{h(\Omega_i)}{|\,\Omega_i\,|}.$$

We assume that $\Gamma = \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus ... \oplus \mathbb{Z}}_{n}$ and $\gamma_1, \gamma_2, .., \gamma_n$ is the canonical basis.

Take k such that $(2k+1)^n > N$. For $i > 5\,k+2$, take $m > 1$ such that $k + m\,(2k+1) \leq i \leq k + (m+1)\,(2\,k+1)$.

Consider

$$\Omega'_{k,i} = \cup \; \{ \; \Omega'_k + (\, j_1(2k+1), ..., j_n(2k+1)\,) \,|\, j_1 \text{ is even }, -m \leq j_1 \leq m-1,$$

$$-m \leq j_2, .., j_n \leq m, \; (j_1, j_2, ..., j_n) \neq (0, ..., 0) \; \}.$$

Then, $\Omega'_{k,i} \subset \Omega_i$, and $\Omega'_{k,i}$ is a finite union of disjoints translates of Ω'_k. Moreover $\lim_{i\to\infty} \frac{|\Omega'_{k,i}|}{|\Omega_i|} = 1$,

$$|\,\Omega'_{k,i}\,| \geq 2\,(2\,k+1)^n > N \text{ and } |\,\Omega_i - \Omega'_{k,i}\,| \geq 2\,(2\,k+1)^n > N \; .$$

From this follows that

$$h(\Omega_i) = h(\Omega'_{k,i} \,\cup\, (\,\Omega_i - \Omega'_{k,i})\,) \geq h(\,\Omega'_{k,i}\,),$$

By the other hand, all translate of Ω'_k has cardinality bigger than N. Therefore,

$$h(\Omega'_{k,i}) \geq \frac{|\,\Omega'_{k,i}\,|}{|\,\Omega'_k\,|}\, h(\Omega'_k).$$

Then,

$$\frac{h(\Omega_i\,)}{|\,\Omega_i\,|} \;\geq\; \frac{h(\Omega'_{k,i}\,)}{|\,\Omega_i\,|} \;\geq\; \frac{1}{|\Omega_i|} \;\frac{|\,\Omega'_{k,i}\,|\, h(\Omega'_k\,)}{|\,\Omega'_k\,|} \;=\; \frac{|\,\Omega'_{k,i}\,|}{|\,\Omega_i\,|} \;\frac{h(\Omega'_k\,)}{|\,\Omega'_k\,|}\,.$$

Now, for a fixed k, taking $i \to \infty$ in the above inequality we get

$$\lim_{i\to\infty} \frac{h(\Omega_i\,)}{|\,\Omega_i\,|} \;\geq\; \frac{h(\Omega'_k\,)}{|\,\Omega'_k\,|}\,.$$

From this follows that

$$\lim_{i\to\infty} \frac{h(\Omega_i\,)}{|\,\Omega_i\,|} \;\geq \limsup_{k\to\infty} \; \frac{h(\Omega'_k\,)}{|\,\Omega'_k\,|}\,.$$

$\square$

6 Properties of $b(c)$

Lemma 27. *There exists $c \in [0,1]$ such that*

$$b(c) \geq \log(\, dim\, H^*(M,\mathbb{R})\,) > 0$$

Proof:

Note that dim $(\,H^*(M)\,) \geq 2$ because dim $M \geq 1$. Let q be the number of connected components of M.

If $|\Omega_i| = m_i$, take $0 = t_0 < t_1 < ... < t_{m_i} = 1$, a partition of $[0,1]$ in m_i intervals of the same size. By Lemma 2

$$H^*(f_{\Omega_i}^{-1}(-\infty, t_{m_i})) = \oplus_{r>0} H^r(M^{\Omega_i}, \mathbb{R}),$$

Denote A_{ij} a supplement of $H^*(f_{\Omega_i}^{-1}(-\infty, t_{j-1}))$ in $H^*(f_{\Omega_i}^{-1}(-\infty, t_j))$, $1 \leq j \leq m_i$. Then,

$$\sum_{j=1}^{m_i} \dim A_{ij} = \dim H^*(f_{\Omega_i}^{-1}(-\infty, t_{m_i})) = \dim H^*(M^{\Omega_i}, \mathbb{R}) - q.$$

Therefore, there exists a certain $A_{i\,j_i} = A_i$, such that,

$$\dim A_i \;\geq\; \frac{(\dim H^*(M,\mathbb{R}))^{m_i} - q}{m_i}.$$

Denote s_i the middle point of $(t_{j_i-1}, t_{j_i}]$ and $\delta_i = \frac{1}{2\,m_i}$.

Then, by definition of $b_i'(s_i, \delta_i) =$ dim A_i.

There exists a subsequence $s_{i_k} \to c \in [0,1]$, when $k \to \infty$.

Given $\delta > 0$, there exists a $K > 0$ such that $\delta_{i_k} < \delta/2$ and $|s_{i_k} - c| < \delta/2$, if $k > K$.

This means $c - \delta < s_{i_k} - \delta_{i_k}$ and $s_{i_k} + \delta_{i_k} < c + \delta$.

From this follows that $b_{i_k}'(c, \delta) \geq b_{i_k}'(s_{i_k}, \delta_{i_k}) =$ dim A_{i_k}.

Finally, we get

$$\frac{\log(\, b_{i_k}'(c,\delta)\,)}{|\, \Omega_{i_k}\,|} \geq \frac{1}{m_{i_k}} \log \frac{(\dim H^*(M,\mathbb{R}))^{m_{i_k}} - q}{m_{i_k}}.$$

Now, taking limit in $k \to \infty$ in the above expression we get

$$b'(c,\delta) \geq \log(\dim\, (H^*(M,\mathbb{R})).$$

$\square$

Lemma 28. *The function $b(c)$ is upper semicontinuous.*

Proof: Suppose c_k, $k \in \mathbb{N}$ is a sequence of points in $[0,1]$ such that, $c_k \to c$.

Given $\epsilon > 0$, take $\delta > 0$, such that, $b'(c,\delta) < b(c) + \epsilon$. There exists a $N > 0$ such that $|c - c_k| < \delta/2$, if $k \geq N$. Then, $c - \delta < c_k - \delta/2$ and $c_k + \delta/2 < c + \delta$, if $k \geq N$.

Then, $b'_i(c,\delta) \geq b'_i(c_k, \delta/2)$, if $k \geq N$, for all $i = 1, 2, 3, ...$

From this follows that $b'(c,\delta) \geq b'(c_k, \delta/2)$. Therefore,

$$b(c) + \epsilon > b'(c,\delta) \geq b'(c_k,\delta/2) \geq b(c_k), \text{ if } k \geq N.$$

Therefore

$$\limsup_{k\to\infty} b(c_k) \leq b(c) + \epsilon,$$

for any $\epsilon > 0$. From this it follows the claim.

□

Lemma 29. *The function $b(c)$ is concave.*

Proof: Consider $0 \leq c_1 < c_2 \leq 1$ and $0 \leq t \leq 1$, we will show that

$$b(\,t\,c_1 + (1-t)\,c_2) \geq t\,b(c_1) + (1-t)\,b(c_2).$$

First we will show the claim for $t = 1/2$. Denote $\tilde{\Omega}_i = \Omega_i + (2\,i+1)\gamma_1$ and $\Omega'_i = \Omega_i \cup \tilde{\Omega}_i$.

By Proposition 13 and Lemma 21 we get:

$$b'_{\Omega'_i}(1/2\,c_1 +\ 1/2\,c_2,\ \delta) \geq b'_{\Omega_i}(c_1,\delta)\,b'_{\tilde{\Omega}_i}(c_2,\delta) = b'_i(c_1,\delta)\,b'_i(c_2,\delta),$$

for all $\delta > 0$.

Now, applying Lemma 26 to $h(\Omega) = \log b'_\Omega(1/2\,c_1 + 1/2\,c_2, \delta)$, we get $b'(1/2\,c_1 + 1/2\,c_2, \delta) \geq 1/2\,b'(\,c_1,\delta)\ +\ 1/2\,b'(\,c_2,\delta)$.

Now, taking $\delta \to 0$, we get $b(1/2\,c_1 + 1/2\,c_2) \geq 1/2b(\,c_1)\ +\ 1/2\,b(\,c_2)$.

The inequality we have to prove is true for a dense set of values of t in $[0,1]$. Then, by Lemma 28 is true for all $t \in [0,1]$.

□

Corollary 30. *The function $b(c)$ is continuous for $c \in [0,]$.*

Proof: This follows from Lemmas 28 and 29.

□

We collect all results we get above in the next theorem.

Theorem 31. *a)* $0 \leq b(c) \leq \epsilon(c) \leq \log($ *number of critical points of* f_0*), for all* $0 \leq c \leq 1$.

b) $b(c)$ *is continuous on* $[0,1]$

c) $b(c)$ *is concave, that is, its graph is always above the cord*

d) $b(c)$ *is not constant equal zero. Moreover, there exists a point* c *where* $b(c) \geq \log$ *(dim* $H^*(M,\mathbb{R})\,) > 0$

7 An example

The next example shows that the item d) in the above theorem can not be improved.

Take $M = S^n$, $n \geq 1$, and a Morse function $f_0 : M \to [0,1]$ which is surjective with only two critical points. Suppose x_- is the minimum and x_+ the maximum of f_0. We will compute $b(c)$ and $\epsilon(c)$.

Take $\Omega \in B(\Gamma)$ with $|\Omega| = m \geq 1$. For each $\Omega' \subset \Omega$ consider the canonical projection $p_{\Omega'} : M^\Omega \to M^{\Omega'}$. Now, take

$$\mu^{\Omega'} = p_{\Omega'}^*(\,[\,M^{\Omega'}\,]\,) \in H^{n\,|\,\Omega'\,|}(M^\Omega,\mathbb{R}),$$

where [ ] represents fundamental class. Then,

$$\{\,\mu^{\Omega'}\ :\ \Omega' \subset \Omega\}$$

is a $\mathbb{R}$-homogeneous basis of $H^*(M^\Omega,\mathbb{R})$.

For $0 \leq d \leq 1$ denote

$$L_d = \{x \in M^\Omega\ :\ f_\Omega(x) < d\,\} \subset M^\Omega.$$

For $x \in M^\Omega$ we denote by x_γ the corresponding coordinate, where $\gamma \in \Gamma$.

Lemma 32. *If* $0 \leq d \leq 1$*, where* d *is not rational, then*

$$\{\mu^{\Omega'}\ :\ |\Omega'| > m\,(1-d)\,\}$$

is a basis of $H^*(L_d)$*.*

Proof: Take $K_d = M^\Omega - L_d$. By Lemma 2

$$H^*(L_d) = \mathrm{Ker}(\,H^*(M^\Omega,\mathbb{R})\ \to\ H^*(K_d,\mathbb{R})\,)\ \text{(natural restriction)}.$$

The claim follows from

1) $H^k(M^\Omega,\mathbb{R}) \to H^k(K_d,\mathbb{R})$ is zero if $k > m\,(1-d)\,n$, and
2) $H^k(M^\Omega,\mathbb{R}) \to H^k(K_d,\mathbb{R})$ is injective if $k < m\,(1-d)\,n$.

Now we prove (1) and (2).

(1) Suppose $\Omega' \subset \Omega$ is such that $\mu^{\Omega'} \in H^k(M^\Omega)$ where $k > m\,(1-d)\,n$. Then, $|\Omega'| > m\,(1-d)$. Suppose

$$F_{\Omega'} = \{x \in M^\Omega \;:\; x_\gamma = x_-\,,\ \text{if}\,\gamma \in \Omega'\}.$$

If $x \in F_{\Omega'}$, then $f_\Omega(x) \le \frac{1}{m}\,(m-|\Omega'|) < d$. Then, $F_{\Omega'} \cap K_d = \emptyset$. This means that: if $x \in K_d \;\to\; x_\gamma \neq x_-$ for some $\gamma \in \Omega'$. Then, $K_d \subset p_{\Omega'}^{-1}(M^{\Omega'} - \{z\}\,)$ where $z_\gamma = x_-$ for all $\gamma \in \Omega'$.

From this follows

$$\mu^{\Omega'} \,|\, K_d \;=\; p_{\Omega'}^{*}\,(\,[\,M^{\Omega'}\,]\,) \;|\; K_d \;= 0,\ \text{because}\,[\,M^{\Omega'}\,] \;|\; (\,[\,M^{\Omega'}\,] \,-\{z\}\,) = 0.$$

(2) Denote $T = \{x \in M^\Omega \;:\; \text{cardinality}(\,\{\gamma \;:\; x_\gamma = x^+\,\}) \;>\; m\,d\,\}$. The set T is closed.

If $x \in T$, then $f_\Omega(x) > \frac{1}{m}\,m\,d = d$. Then, $T \subset K_d$.

We have to show that

$$H^k(M^\Omega,\mathbb{R}) \to H^k(T,\mathbb{R}) \text{ is injective if } k < m\,(1-d)\,n.$$

As we had seen before $H^k(M^\Omega,\mathbb{R}) = 0$ if k is not multiple of n. Then, we can assume that $k = q\,n$, if $q = 0,1,2,....$ The claim follows from the next lemma, taking s the integer part of $m\,d$, by the exact sequence of homology, given that $U = U_s(\Omega)$.

Lemma 33. *Suppose $s = 0,1,2,..,m$. Suppose*

$$U_s(\Omega) \;=\; \{x \in M^\Omega \;:\; \mathit{card}\,(\,\{\gamma \;:\; x_\gamma = x^+\}\,) \;\le s\},$$

then, $H^k_c(U_s(\Omega),\mathbb{R}) = 0$, if $k < (m-s)\,n$.

Proof: The claim is trivial for $s = 0$ or $s = m$ ($U_0(\Omega)$ is homeomorphic to $(\,\mathbb{R}^n\,)^m$).

The proof is by induction in m. The claim for $m = 1$ is trivial. Suppose is true for $m-1 \ge 1$. Take $0 < s < m$. Fix $w \in \Omega$ and take $\Omega' = \Omega - \{w\}$.

Consider $\varphi : M^{\Omega'} \to M^\Omega$ and $\psi : M^{\Omega'} \times (M - \{\,x^+\,\}) \;\to\; M^\Omega$, where for a given x we define $\varphi(x)$ by $x_\omega = x^+$ if $x \in M^{\Omega'}$, and $\psi(x,u)$ is defined by $x_w = u$ if $x \in M^{\Omega'}$ and $u \in M$, $u \neq x^+$.

ψ identifies $U_s(\Omega') \times (\,M - \{x^+\}\,)$ with an open set A contained in $U_s(\Omega)$.

Moreover, φ identifies $U_{s-1}(\Omega')$ with the complement of this open set A in $U_s(\Omega)$.

As $M - \{x^+\}$ is homeomorphic to $\mathbb{R}^n$ and by recurrence we get that

$$H_c^k(\, U_s(\Omega') \times (\, M - \{x^+\}\,)\,, \mathbb{R}) = 0,$$

if $k < (m-1-s)\,n + n = (m-s)\,n$ and, moreover, $H_c^k(\, U_{s-1}(\Omega', \mathbb{R})\,) = 0$, if $k < (\,(m-1) - (s-1)\,)\,n = (m-s)\,n$.

The exact sequence of homology finish the proof.

$\square$

Now we fix irrationals d_1, d_2, $0 < d_1 < d_2 < 1$. Denote $a_m = m\,(1-d_1)$, $b_m = m(1-d_2)$, and, $c_m = \dim\,(\, H^*(L_{d_2})/H^*(L_{d_1})\,)$.

By Lemma 32 we get

$$c_m = \sum \{ \begin{pmatrix} m \\ j \end{pmatrix} : \, b_m < j < a_m \}.$$

Assume m is much more bigger than $(d_2 - d_1)$.

Take an integer j_m, such that $b_m < j_m < a_m$,

$$\begin{pmatrix} m \\ j_m \end{pmatrix} = \sup \{ \begin{pmatrix} m \\ j \end{pmatrix} : \, b_m < j < a_m \}.$$

Then,

$$\begin{pmatrix} m \\ j_m \end{pmatrix} \le c_m \le (a_m - b_m + 1) \begin{pmatrix} m \\ j_m \end{pmatrix}.$$

By Stirling formula:

$$\frac{1}{m} \log \begin{pmatrix} m \\ j \end{pmatrix} \sim \frac{1}{m} \log(\frac{m^{m+\ 1/2}}{j^{j+\ 1/2}\,(m-j)^{\,m-j+\ 1/2}}) =$$

$$\frac{1}{m} \log(\, m^{-1/2}\,(\frac{j}{m})^{-1/2}\,(1-\frac{j}{m})^{-1/2}\,(\frac{j}{m})^{-j}\,(1-\frac{j}{m})^{-m+j}\,).$$

Therefore,

$$\frac{1}{m} \log \begin{pmatrix} m \\ j_m \end{pmatrix} \sim \frac{1}{m} \log(\, (\frac{j_m}{m})^{-j_m}\,(1-\frac{j_m}{m})^{-m+j_m}\,) =$$

$$-\frac{j_m}{m}\, \log(\frac{j_m}{m}) \, - \, (1-\frac{j_m}{m})\, \log(1-\frac{j_m}{m}),$$

when $m \sim \infty$.

As $1-d_2 < \frac{j_m}{m} < 1-d_1$, then (changing x by $(1-x)$) we get

$$\limsup_{m\to\infty} \frac{1}{m} \log \binom{m}{j_m} \leq \sup_{d_1<x<d_2} (\,-x\log(x) - (1-x)\,\log(1-x)\,),$$

and

$$\liminf_{m\to\infty} \frac{1}{m} \log \binom{m}{j_m} \geq \inf_{d_1<x<d_2} (\,-x\log(x) - (1-x)\,\log(1-x)\,).$$

From this follows

$$\limsup_{m\to\infty} \frac{\log c_m}{m} \leq \sup_{d_1<x<d_2} (\,-x\log(x) - (1-x)\,\log(1-x)\,),$$

and

$$\liminf_{m\to\infty} \frac{\log c_m}{m} \geq \inf_{d_1<x<d_2} (\,-x\log(x) - (1-x)\,\log(1-x)\,).$$

Proposition 34.

$$\epsilon(c) = b(c) = -\,c\,\log c - \,(1-c)\,\log(1-c),\;\; 0 \leq c \leq 1.$$

Proof:
Given $0 < c < 1$, there exists small $\delta > 0$ such that

$$0 < c-\delta < c < c+\delta < 1 \;\text{ and }\; c-\delta, c+\delta \text{ are not in } \mathbb{Q}.$$

From the above for $d_1 = c-\delta$ and $d_2 = c+\delta$ we get

$$\inf_{d_1<x<d_2} (\,-x\log(x) - (1-x)\,\log(1-x)\,) \leq b'(c,\delta) \leq$$

$$\sup_{d_1<x<d_2} (\,-x\log(x) - (1-x)\,\log(1-x)\,).$$

Now, taking $\delta \to 0$, we get

$$b(c) \;=\; (\,-c\log(c) - (1-c)\,\log(1-c)\,).$$

For $c=0$ or $c=1$ the result follows from continuity.
Now we will estimate $\epsilon(c)$.
The critical values of f_Ω are $0, \frac{1}{m}, \frac{2}{m}, ..., 1$.

To the critical values $\frac{j}{m}$ $(j = 0,1,2,..,m)$ corresponds $\binom{m}{j}$ critical points.

Therefore, given $d_1, d_2 \in \mathbb{R}$ $d_1 < d_2$, the number c'_m of critical points of f_Ω in $f_\Omega^{-1}(d_1, d_2)$ is

$$c'_m = \sum \{ \begin{pmatrix} m \\ j \end{pmatrix} \ : \ d_1 < \frac{j}{m} < d_2 \} =$$

$$\sum \{ \begin{pmatrix} m \\ j \end{pmatrix} \ : \ m\,(1 - d_2) < \ j \ < m\,(1 - d_1) \}.$$

The computation of $\epsilon(c)$ is analogous to the one for $b(c)$. This also follows from the last Theorem and the fact that $H^*(M) =$ number critical points of f_0 in the present case.

□

8 About the definition of $b(c)$

We will show that the definition of $b(c)$ presented here coincides with the one in [7].

First we need some preliminary results.

Suppose X is a compact connected oriented C^∞ manifold.

Lemma 35. *Given an open set V in X consider $\alpha \in H^*(X, \mathbb{R})$ such that $\alpha|_V \neq 0$. Then, there exists $\beta \in H^*(V)$ such that $\alpha \wedge \beta \neq 0$.*

Proof: Take $w \in \alpha$. As $\alpha|_V \neq 0$, then there exists a cycle z on V such that $\int_z w \neq 0$.

Suppose w' is a closed form with compact support on V such that its cohomology class in $H_c^*(V, \mathbb{R})$ is the Poincare dual of the homology class of z in $H_*(V, \mathbb{R})$.

w' can be extended to a closed form on X (putting 0 where needed) and by Poincare duality:

$$0 \neq \int_z w \ = \int_V w \wedge w' \ = \int_X w \ \wedge w'.$$

Therefore, $w \wedge w'$ is not exact on X.

Denote $\beta \in H^*(X, \mathbb{R})$ the cohomology class of w'. By Lemma 2 we have that $\beta \in H^*(V)$. As $w \wedge w'$ is not exact we get that $\alpha \wedge \beta \neq 0$.

□

Notation: if $S \subset X$, then $\mathcal{H}^*(S) = \cap \{ H^*(W) \ : W \subset X$ is an open set and $S \subset W \}$.

Lemma 36. *Suppose $U, V \subset X$ are open sets and $X = U \cup V$. Take $K = U - V$ and $\alpha \in H^*(U)$. Then, $\alpha \wedge \beta = 0$ for all $\beta \in H^*(V)$, if and only if, $\alpha \in \mathcal{H}^*(K)$.*

Proof: Suppose $\alpha \in \mathcal{H}^*(K)$ and take $\beta \in H^*(V)$. By Lemma 8 there exists $w \in \beta$ such that supp $w \subset V$.

Take $W = X-$ supp w (which contains K). By definition we get that $\alpha \in H^*(W)$. Then, by Lemma 8, there exists $w' \in \alpha$ such that supp $w' \subset W$. Therefore, $w \wedge w' = 0$, and finally it follows that $\alpha \wedge \beta = 0$.

Reciprocally, suppose that $\alpha \wedge \beta = 0$ for all $\beta \in H^*(V)$. By Lemma 35 we have that $\alpha | V = 0$. Take $W \supset K$, then $V \cup W = X$. Therefore, by definition $\alpha \in H^*(W)$.

$\square$

Lemma 37. *Take $K \subset X$ a compact submanifold with boundary such that $K - \delta K$ is an open subset of X.*

Then,

$$\mathcal{H}(K) = \textit{ Ker } (\, H^*(X, \mathbb{R}) \to H^*(X - K, \mathbb{R})\,) \textit{ restriction.}$$

Proof: Take W an open set by adding a necklace to K. Then, $X - K$ can be retracted by deformation over $X - W$.

Then, if $\alpha \in H^*(X, \mathbb{R})$, we get that $\alpha|_{X-K} = 0$ is equivalent to $\alpha|_{X-W} = 0$.

Now, the claim follows from Lemma 2 and by the definition of $\mathcal{H}(K)$. $\square$

Corollary 38. *Under the same hypothesis of last lemma it also follows that $\mathcal{H}(K) = H^*(\textit{ int } (K)\,)$.*

Proof: This follows from the fact that $H^*(X-\text{ int } (K)\,, \mathbb{R}) \to H^*(X - K, \mathbb{R})$ is an isomorphism.

$\square$

Proposition 39. *Suppose U, V are open sets such that $X = U \cup V$ and moreover that $\overline{U}, \overline{V}$ are submanifolds with boundary of X.*

Consider the linear transformation L such that

$$L : H^*(U) \to \textit{Hom}\,(\, H^*(V), H^*(U \cap V)\,),$$

where, $a \to (\, b \to a \wedge b\,)$.

Then, the rank of L is dim $(\, H^*(U)/H^*(M - \overline{V})\,)$.

Proof: By Lemma 36 we get that Ker $L = H^*(X - V)$. Finally, by the last corollary $H^*(X - V) = H^*(M - \overline{V})$.

$\square$

Consider now a Morse function $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$, $\delta > 0$.

Definition 40. *$b_{c,\delta}(f)$ is the rank of the linear transformation*

$$H^*(\, f^{-1}(-\infty, c+\delta)\,) \to Hom\,(\, H^*(f^{-1}(\, c-\delta, \infty\,), H^*(f^{-1}(c-\delta, c+\delta)\,)\,),$$

where $a \to (b \to a \wedge b)$.

Note that $b_{c,\delta}(f)$ decreases with δ.

Lemma 41. *If $c - \delta$ and $c + \delta$ are regular values of f, then*

$$b_{c,\delta}(f) = b'_{c,\delta}(f).$$

Proof: Just apply Proposition 39 to $U = f^{-1}(-\infty, c+\delta)$ and $V = f^{-1}(\, c-\delta, \infty\,)$.

$\square$

Note that $b_\Omega(c, \delta) = b_{c,\delta}(f_\Omega)$, where $\Omega \in B(\Gamma)$ and $\Omega \neq \emptyset$, and moreover that $b_i(c, \delta) = b_{\Omega_i}(c, \delta)$. The next limit exists (see [7]).

Definition 42.

$$b(c, \delta) = \lim_{i \to \infty} \frac{\log(\, b_i(c, \delta)\,)}{|\, \Omega_i \,|}.$$

The set $S \subset [0, 1]$ of all critical values of all f_Ω is countable. By Lemma 41 we get that $b'_i(c, \delta) = b_i(c, \delta)$ if $c - \delta \notin S$ and $c + \delta \notin S$. Therefore, $b'(c, \delta) = b(c, \delta)$ if $c - \delta \notin S$ and $c + \delta \notin S$.

Finally,

$$\lim_{\delta \to 0} b'(c, \delta) = \lim_{\delta \to 0} b(c, \delta)$$

because both limits exist.

Therefore the function $b(c)$ we define coincides with the one presented in [7].

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References

[1] M. Asaoka, T. Fukaya, K. Mitsui and M. Tsukamoto, Growth of critical points in one-dimensional lattice systems, preprint Arxiv 2012.

[2] A. T. Baraviera, L. Cioletti, A. O. Lopes, J. Mohr and R. R. Souza, On the general one-dimensional XY model: positive and zero temperature, selection and non-selection. *Rev. Math. Phys.* 23,no. 10, 1063-1113, 82Bxx, 2011.

[3] A. Bovier. Statistical Mechanics of Disordered Systems. A Mathematical Perspective. Cambridge University Press (2006).

[4] A. Baraviera, R. Leplaideur and A. O. Lopes, Ergodic Optimization, Zero temperature limits and the Max-Plus Algebra, mini-course in XXIX Colóquio Brasileiro de Matemática - IMPA - Rio de Janeiro (2013)

[5] A. O. Lopes, J. Mohr, R. R. Souza and P. Thieullen, Negative Entropy, Zero temperature and stationary Markov chains on the interval, *Bull. Soc. Bras. Math.* Vol 40 n 1, (2009), 1-52.

[6] A. O. Lopes, J. K. Mengue, J. Mohr and R. R. Souza, Entropy and Variational Principle for one-dimensional Lattice Systems with a general a-priori probability: positive and zero temperature" to appear in *Erg. Theo. and Dyn. Syst.*

[7] M. Bertelson and M. Gromov, Dynamical Morse Entropy, Modern dynamical systems and applications, 27-44, Cambridge Univ. Press, Cambridge (2004)

[8] R.B. Israel, Convexity in the theory of lattice gases, Princeton University Press, 1979.

[9] W. Chou and R. Griffiths, *Ground states of one-dimensional systems using effective potentials*, Physical Review B, Vol. 34, N 9, 6219-6234, 1986

[10] D. Coronel and J. Rivera-Letelier, Sensitive dependence of Gibbs measures, preprint (2014)

[11] L. Cioletti and A. Lopes, Interactions, Specifications, DLR probabilities and the Ruelle Operator in the One-Dimensional Lattice

[12] L. Cioletti and A. Lopes, Phase Transitions in One-dimensional Translation Invariant Systems: a Ruelle Operator Approach, to appear in Journ. of Stat. Phys.

[13] Y. Fukui and M. Horiguchi, One-dimensional Chiral XY Model at finite temperature, Interdisciplinary Information Sciences, Vol 1, 133-149, N. 2 (1995)

[14] T. Fukaya and M. Tsukamoto, Asymptotic distribution of critical values. Geom. Dedicata 143 (2009), 6367

[15] W. Massey, Homology and Cohomology, M. Dekker (1978)

[16] J. Milnor, Morse Theory, Princeton University Press, Princeton (1963)

[17] B. Simon, The Statistical Mechanics of Lattice Gases, Princeton Univ Press, 1993

[18] O. Sarig, Lecture notes on thermodynamic formalism for topological Markov shifts, *Penn State*, 2009.

[19] A. C. D. van Enter and W. M. Ruszel. Chaotic Temperature Dependence at Zero Temperature. *Journal of Statistical Physics*. Vol. 127. No. 3. (2007), 567-573.

[20] C. Thompson. *Infinite-Spin Ising Model in one dimension*. Journal of Mathematical Physics. (9): N.2 241-245, 1968.